

expint: Exponential integral and incomplete gamma function

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1 Introduction

The exponential integral

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt, \quad x \in \mathbb{R}$$

and the incomplete gamma function

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt, \quad x > 0, \quad a \in \mathbb{R}$$

are two closely related functions that arise in various fields of mathematics.

expint is a small package that intends to fill a gap in R's support for mathematical functions by providing facilities to compute the exponential integral and the incomplete gamma function. Furthermore, and perhaps most conveniently for R package developers, the package also gives easy access to the underlying C workhorses through an API. The C routines are derived from the GNU Scientific Library (GSL; [Galassi et al., 2009](#)).

Package **expint** started its life in version 2.0-0 of package **actuar** ([Dutang et al., 2008](#)) where we extended the range of admissible values in the computation of limited expected value functions. This required an incomplete gamma function that accepts negative values of argument a , as explained at the beginning of Appendix A of [Klugman et al. \(2012\)](#).

2 Exponential integral

[Abramowitz and Stegun \(1972, Section 5.1\)](#) first define the exponential integral as

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt. \tag{1}$$

An alternative definition (to be understood in terms of the Cauchy principal value due to the singularity of the integrand at zero) is

$$\text{Ei}(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt, \quad x > 0.$$

The above two definitions are related as follows:

$$E_1(-x) = -\text{Ei}(x), \quad x > 0. \quad (2)$$

The exponential integral can also be generalized to

$$E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n} dt, \quad n = 0, 1, 2, \dots, \quad x > 0,$$

where n is then the *order* of the integral. The latter expression is closely related to the incomplete gamma function ([section 3](#)) as follows:

$$E_n(x) = x^{n-1} \Gamma(1-n, x). \quad (3)$$

One should note that the first argument of function Γ is negative for $n > 1$.

The following recurrence relation holds between exponential integrals of successive orders:

$$E_{n+1}(x) = \frac{1}{n} [e^{-x} - x E_n(x)]. \quad (4)$$

Finally, $E_n(x)$ has the following asymptotic expansion:

$$E_n(x) \asymp \frac{e^{-x}}{x} \left(1 - \frac{n}{x} + \frac{n(n+1)}{x^2} - \frac{n(n+1)(n+2)}{x^3} + \dots \right). \quad (5)$$

3 Incomplete gamma function

From a probability theory perspective, the incomplete gamma function is usually defined as

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt, \quad x > 0, \quad a > 0.$$

Function `pgamma` already implements this function in R (just note the differing order of the arguments).

Now, the definition of the incomplete gamma function of interest for this package is rather the following ([Abramowitz and Stegun, 1972](#), Section 6.5):

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt, \quad x > 0, \quad a \in \mathbb{R}. \quad (6)$$

Note that a can be negative with this definition. Of course, for $a > 0$ one has

$$\Gamma(a, x) = \Gamma(a) [1 - P(a, x)]. \quad (7)$$

Integration by parts of the integral in (6) yields the relation

$$\Gamma(a, x) = -\frac{x^a e^{-x}}{a} + \frac{1}{a} \Gamma(a+1, x).$$

When $a < 0$, this relation can be used repeatedly k times until $a+k$ is a positive number. The right hand side can then be evaluated with (7). If $a = 0, -1, -2, \dots$, this calculation requires the value of

$$G(0, x) = \int_x^\infty \frac{e^{-t}}{t} dt = E_1(x),$$

the exponential integral defined in (1).

4 R interfaces

Package **expint** provides one main and four auxiliary R functions to compute the exponential integral, and one function to compute the incomplete gamma function. The synopses of these are the following:

```
expint(x, order = 1L, scale = FALSE)
expint_E1(x, scale = FALSE)
expint_E2(x, scale = FALSE)
expint_En(x, order, scale = FALSE)
expint_Ei(x, scale = FALSE)
gammainc(a, x)
```

Let us first go over function `gammainc` since there is less to discuss. The function takes in argument two vectors or real numbers (non-negative for argument x) and returns the value of $\Gamma(a, x)$. The function is vectorized in arguments a and x , so it works similar to, say, `pgamma`.

We now turn to the `expint` family of functions. Function `expint` is a unified interface to compute exponential integrals $E_n(x)$ of any (non-negative) order, with default the most common case $E_1(x)$. The function is vectorized in arguments x and `order`. In other words, one can compute the exponential integral of a different order for each value of x .

```
> expint(c(1.275, 10, 12.3), order = 1:3)
[1] 1.408099e-01 3.830240e-06 3.009983e-07
```

Argument `order` should be a vector of integers. Non-integer values are silently coerced to integers using truncation towards zero.

When argument `scale` is `TRUE`, the result is scaled by e^x .

Functions `expint_E1`, `expint_E2` and `expint_En` are simpler, slightly faster ways to directly compute exponential integrals $E_1(x)$, $E_2(x)$ and $E_n(x)$, the latter for a *single* order n (the first value of `order` if `order` is a vector).

```
> expint_E1(1.275)
[1] 0.1408099
```

```
> expint_E2(10)
[1] 3.83024e-06
> expint_En(12.3, order = 3L)
[1] 3.009983e-07
```

Finally, function `expint_Ei` is provided as a convenience to compute $Ei(x)$ using (2).

```
> expint_Ei(5)
[1] 40.18528
> -expint_E1(-5)      # same
[1] 40.18528
```

5 Accessing the C routines

The actual workhorses behind the R functions of [section 4](#) are C routines with the following prototypes:

```
double expint_E1(double x, int scale);
double expint_E2(double x, int scale);
double expint_En(double x, int order, int scale);
double gamma_inc(double a, double x);
```

Package **expint** makes these routines available to other packages through declarations in the header file ‘include/expintAPI.h’ in the package installation directory. The developer of some package **pkg** who wants to use a routine — say `expint_E1` — in her code should proceed as follows.

1. Add package **expint** to the Imports and LinkingTo directives of the ‘DESCRIPTION’ file of **pkg**;
2. Add an entry ‘import(expint)’ in the ‘NAMESPACE’ file of **pkg**;
3. Declare a native routine that will call `expint_E1` in a header file of **pkg**. For example, one could declare a routine named `pkg_expint_E1` (to avoid any name clash) in the ‘src/pkg.h’ file as follows:

```
double(*pkg_expint_E1)(double,int);
```

4. Define the aforementioned routine with a call to `R_GetCCallable` in the initialization routine `R_init_pkg` of **pkg** ([R Core Team, 2016](#), Section 5.4). For the current example, the file ‘src/init.c’ of **pkg** would contain the following code:

```
void R_init_pkg(DllInfo *dll)
{
    R_registerRoutines( /* native routine registration */ );
```

```

pkg_expint_E1 = (double (*)(double,int,int))
                R_GetCCallable("expint", "expint_E1");
}

```

5. Include the package header file 'pkg.h' in any C file making use of routine `pkg_expint_E1`.

To help developers get started, **expint** ships with a complete test package implementing the above; see the 'example_API' sub-directory in the installation directory. This test package uses the .External R to C interface and, as a bonus, shows how to vectorize an R function on the C side (the code for this being mostly derived from base R).

6 Implementation details

As already stated, the C routines mentioned in [section 5](#) are derived from code in the GNU Scientific Library ([Galassi et al., 2009](#)).

For exponential integrals, the main routine `expint_E1` computes $E_1(x)$ using Chebyshev expansions ([Gil et al., 2007](#), chapter 3). Routine `expint_E2` computes $E_2(x)$ using `expint_E1` with relation (4) for $x < 100$, and using the asymptotic expression (5) otherwise. Routine `expint_En` simply relies on `gamma_inc` to compute $E_n(x)$ for $n > 2$ through relation (3).

For the sake of providing routines that better fit within the R ecosystem and coding style, we made the following changes to the original GSL code:

1. routines now compute a single value and return their result by value;
2. accordingly, calculation of the approximation error was dropped in all routines;
3. most importantly, `gamma_inc` does not compute $\Gamma(a, x)$ for $a > 0$ with (7) using the GSL routines, but rather using routines `gammafn` and `pgamma` part of the R API.

The following illustrates the last point.

```

> options(digits = 20)
> gammainc(1.2, 3)
[1] 0.065421428091009217742
> gamma(1.2) * pgamma(3, 1.2, 1, lower = FALSE)
[1] 0.065421428091009217742

```

7 Alternative packages

The Comprehensive R Archive Network¹ (CRAN) contains a number of packages with features overlapping **expint**. We review the similarities and differences here.

The closest package in functionality is **gsl** (Hankin, 2006). This package is an R wrapper for the special functions and quasi random number generators of the GNU Scientific Library. As such, it provides access to basically the same C code as used in **expint**. Apart from the changes to the GSL code mentioned in section 6, the main difference between the two packages is that **gsl** requires that the GSL be installed on one's system, whereas **expint** is a regular, free standing R package.

Package **VGAM** (Yee, 2015) is a large, high quality package that provides functions to compute the exponential integral $Ei(x)$ for real values, as well as $e^{-x} Ei(x)$ and $E_1(x)$ and their derivatives (up to the third derivative). Functions `expint`, `expexpint` and `expint.E1` are wrappers to the Netlib² FORTRAN subroutines in file `ei.f`. **VGAM** does not provide an API to its C routines.

Package **pracma** (Borchers, 2016) provides a large number of functions from numerical analysis, linear algebra, numerical optimization, differential equations and special functions. Its versions of `expint`, `expint.E1`, `expint.Ei` and `gammainc` are entirely written in R with perhaps less focus on numerical accuracy than the GSL and Netlib implementations. The version of `gammainc` only supports positive values of a .

Package **frmqa** (Tran, 2012) has a function `gamma_inc_err` that computes the incomplete gamma function using the incomplete Laplace integral, but it is only valid for $a = j + \frac{1}{2}$, $j = 0, 1, 2, \dots$

Package **zipfR** (Evert and Baroni, 2007) introduces a set of functions to compute various quantities related to the gamma and incomplete gamma functions, but these are essentially wrappers around the base R functions `gamma` and `pgamma` with no new functionalities.

8 Examples

We tabulate the values of $E_n(x)$ for $x = 1.275, 10, 12.3$ and $n = 1, 2, \dots, 10$ as found in examples 4–6 of Abramowitz and Stegun (1972, section 5.3).

```
> x <- c(1.275, 10, 12.3)
> n <- 1:10
> structure(t(outer(x, n, expint)),
+           dimnames = list(n, paste("x =", x)))
      x = 1.275      x = 10      x = 12.3
1 0.14080993 4.156969e-06 3.439534e-07
2 0.09989831 3.830240e-06 3.211177e-07
```

¹<https://cran.r-project.org>

²<http://www.netlib.org>

```

3 0.07603031 3.548763e-06 3.009983e-07
4 0.06083077 3.304101e-06 2.831550e-07
5 0.05046793 3.089729e-06 2.672346e-07
6 0.04301687 2.900528e-06 2.529517e-07
7 0.03743074 2.732441e-06 2.400730e-07
8 0.03310097 2.582217e-06 2.284066e-07
9 0.02965340 2.447221e-06 2.177930e-07
10 0.02684699 2.325303e-06 2.080990e-07

```

We also tabulate the values of $\Gamma(a, x)$ for $a = -1.5, -1, -0.5, 1$ and $x = 1, 2, \dots, 10$.

```

> a <- c(-1.5, -1, -0.5, 1)
> x <- 1:10
> structure(t(outer(a, x, gammamainc)),
+           dimnames = list(x, paste("a =", a)))
      a = -1.5      a = -1      a = -0.5      a = 1
1 1.264878e-01 1.484955e-01 1.781477e-01 3.678794e-01
2 1.183299e-02 1.876713e-02 3.009876e-02 1.353353e-01
3 1.870260e-03 3.547308e-03 6.776136e-03 4.978707e-02
4 3.706365e-04 7.995573e-04 1.733500e-03 1.831564e-02
5 8.350921e-05 1.992938e-04 4.773965e-04 6.737947e-03
6 2.045031e-05 5.304291e-05 1.379823e-04 2.478752e-03
7 5.310564e-06 1.478712e-05 4.127115e-05 9.118820e-04
8 1.440569e-06 4.267206e-06 1.266464e-05 3.354626e-04
9 4.042025e-07 1.264846e-06 3.964430e-06 1.234098e-04
10 1.165117e-07 3.830240e-07 1.260904e-06 4.539993e-05

```

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